Effective Lengths of Columns in Braced Multi-Storey Frames†

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Günay ÖZMEN**

ABSTRACT

In several design codes and specifications, simplified formulae, nomographs or charts are given for determining the effective lengths of frame columns. It is shown that these simplified approaches may yield rather erroneous results in most of the cases. This is due to the fact that, the code formulae utilize only local stiffness distributions, thus ignoring the general behaviour of the system. In this paper, a simplified procedure for determining approximate values for the effective lengths of braced multi-storey frame columns is developed. The procedure utilizes a simple average calculation and yields errors less than 10 %, which may be considered suitable for practical purposes. The proposed procedure is applied to several numerical examples and it is shown that all the errors are in the acceptable range.

Key words: buckling load, buckling length, effective length, non-sway mode, braced frames, isolated subassembly, multi-storey frames, design codes.

1. INTRODUCTION

Determining the effective (buckling) lengths of frame columns is one of the significant phases of frame design. Theoretically, effective length of an individual column is determined by calculating the system-buckling load of the frame. Since a full system instability analysis, may be quite cumbersome for frames encountered in practical applications, simplified formulae, nomographs or charts are given for determining the effective lengths of frame columns in most of the design codes and specifications. The design codes currently used in Turkey utilize either nomographs (TS 648) or formulae (TS 500) which are adopted from AISC (1988) and ACI (1989), respectively, [1], [2], [3], [4]. The simplified formulae, nomographs and charts of the design codes are based on the so-called “Isolated subassembly approach” which has been originally developed by Galambos, [5]. This approach assumes the individual frame columns to be independent of each other and the effective lengths are computed as a function of stiffness distributions at upper and lower ends. Similar assumptions are adopted in other widely used contemporary design codes, [6], [7]. The major defect of the methods based on isolated subassembly approach is that they do not properly recognize the interaction of adjacent elements other than the ones at immediate neighbourhood of the joints. Hellesland and Bjorhovde have shown that this

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† Published in Teknik Dergi Vol. 19, No. 1 January 2008, pp: 4333-4346
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approach may result in significant errors in certain cases, [8]. Several investigations have been carried out in order to improve the applicability of the subassembly approach, [9] ~ [21]. The majority of the studies to improve the results of the subassembly approach are devoted to “unbraced” frames. On the other hand, for the case of “braced” frames a limited number of studies exist, whereby several methods are developed for the purpose of improving the code oriented results. Aristizabal-Ochoa and Cheong-Siat-Moy have developed general formulations including solutions for both braced, unbraced and “partially braced” structures, [16], [17]. Another interesting improvement approach is proposed by Hellesland and Bjorhovde which involves a post processing procedure using weighted mean values of effective lengths, [20]. In general it is found that this method provides insignificant errors on the unsafe side. It is recommended that extended numerical examples are needed to establish the validity of this method. Mahini and Seyyedian have proposed another post processing approach depending on determination of the critical elements of the structure, [22].

It is observed that, the general trend in recent developments concerning the design codes is to abandon the “Isolated subassembly approach” completely. As a matter of fact, in the 1999 edition of AISC, this approach is discarded and it is stated that “…the effective length factor K of compression members shall be determined by structural analysis”, [23]. However, it is observed that the isolated subassembly approach is still adopted in several widely used design codes, [1], [7] and [24].

In this study, a practical method is developed for determining the effective lengths of columns of braced frames. This method is applied by computing a simple average value using the results obtained from the formulae given in design codes. It is found that the proposed method has a wide range of application and produce errors which are within the acceptable range.

2. BUCKLING LOADS OF BRACED MULTI-STOREY FRAMES

A multi-storey braced frame which is composed of beams and columns made of linear elastic material is under the effect of vertical loads as shown in Fig. 1a.

The frame is in the state of “Stable Equilibrium” and, if the axial deformations are neglected, all the displacements and deformations are zero. Internal forces of the frame columns consist of only axial forces \( N_{ij} \) while all the internal forces of beams are zero. Each axial load may be expressed as

\[
N_{ij} = n_i P
\]  

(1)

where \( n_i \) is a dimensionless parameter and \( P \) is an arbitrarily chosen load parameter. When the load parameter reaches a critical \( P_c \) value, another state of “Unstable Equilibrium” may exist. The displacement diagram corresponding to this new state, which is shown schematically in Fig. 1b, is called the “Buckling Mode” of the structure. Once the buckling
load parameter $P_k$ is determined, the effective length $s_{ij}$ of an individual column can be computed by

$$h_{k,ij} = \pi \frac{EI_{ij}}{n_y P_k} \sqrt{2}$$

where $EI_{ij}$ is the bending stiffness of the column, [25], [26].

In certain simple cases with a small number of elements, buckling load parameter may be determined by using the so-called “stability functions” or “second order stiffness coefficients”, [25], [26], [27]. For general cases (for frames with large number of elements) however, it is necessary to utilise specially prepared software, [28]. In cases where multi-purpose software such as SAP2000 is used, it is necessary to split the columns into smaller parts in order to obtain sufficiently rigorous values for buckling load values, [29].

In this paper, a practical method for determining the approximate values for buckling loads will be explained and then applied to numerical examples. The proposed method is based on computing the buckling loads by means of a simple average whereby the effective lengths given by the design codes are used.

3. EFFECTIVE LENGTHS ACCORDING TO DESIGN CODES

In several design codes and specifications, simplified formulae, nomographs or charts are given for calculating the buckling (effective) lengths of individual columns. Thus it has been possible to liberate the designer from applying the tedious computations (or special software) which is necessary for the calculation of the overall-buckling load. In the
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following, the computational principals specified in the Turkish Steel Structures Code - TS 648 “Specifications for Design and Construction of Steel Structures” will be summarized.

As stated in Article “3.2 – Elements Subjected to Axial Force” of TS 648, first the so called “Distribution coefficients” are computed by

$$G = \frac{\sum L_c h_c}{\sum L_g}.$$  \hspace{1cm} (3)

at both ends of the column. Here $I_c$, $h_c$, $I_g$ and $L_g$ denote column moment of inertia, column length, beam moment of inertia and beam length, respectively. Once the coefficients $G_A$ and $G_B$ at upper and lower ends of the column are computed by using Eq. (3), it is necessary to solve the Galambos equation

$$f(x) = \frac{G_A G_B}{4} x^2 + \frac{G_A + G_B}{2} \left(1 - \frac{x}{\tan x}\right) + \frac{\tan x}{x} = 0$$ \hspace{1cm} (4)

for variable $x$, [5]. Then the effective length $h_k$ is computed by

$$K = \frac{\pi}{x}$$ \hspace{1cm} (5)

$$h_k = K h_c.$$ \hspace{1cm} (6)

In order to liberate the designer from solving Eq. (4), nomographs for both braced and unbraced frames are provided in TS 648. Discussion of effective lengths of unbraced frames is left out of the scope of this study. The nomograph given in TS 648 for braced frames is shown in Fig. 2.

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Thus the effective length factor $K$ can easily be taken from this nomograph by using the coefficients $G_A$ and $G_B$ without needing to solve the related trigonometric equation.

It must be pointed out that this same nomograph was given in earlier editions of TS 500 for reinforced concrete frames. Recently, it is replaced by the simplified formula:

$$K = 0.7 + 0.05(G_A + G_B) \leq 0.85 + 0.05G_{\text{min}} \leq 1.0$$

(7)

for computing the approximate value of the effective length factor $K$. It must be noted that cracked sections are to be used in computing the moment of inertia values for reinforced concrete structures.
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In Eurocode 3 (DIN 18800), the distribution coefficients are computed in a slightly different manner and charts are provided for determining the effective length factors, [6]. The details of this code are not included herein for the sake of brevity.

It must be pointed out that all values for effective length factor $K$ vary between 0.5 and 1.0 for all the specifications considered in this study. However, Bridge and Fraser have shown that the values of $K$ may exceed 1.0 in certain cases of braced frames, [11]. In the following, it will be shown on numerical examples that rather large errors may be encountered by using the formulae (or nomographs or charts) given in codes.

### 3.1 Typical Frames

With the purpose of testing the formulae and nomographs given in various codes, six “Typical frames” shown in Fig. 3 are chosen.

The exact values of the buckling loads for the typical frames are determined by using a special software prepared by Girgin, [28]. In this software, second order stiffness matrices of elements are considered and a successive approximation procedure is used for the application of the determinant criterion to the system stiffness matrix. It is possible to express the exact buckling load values of all the considered frames as:

$$P_k = C_k \frac{EI}{h^2}$$  \hspace{1cm} (8)

where the multipliers $C_k$ obtained for typical frames are as shown in Table 1.

<table>
<thead>
<tr>
<th>Type</th>
<th>$C_k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>AS</td>
<td>14.39</td>
</tr>
<tr>
<td>AD</td>
<td>4.22</td>
</tr>
<tr>
<td>BS</td>
<td>21.34</td>
</tr>
<tr>
<td>BD</td>
<td>9.92</td>
</tr>
<tr>
<td>CS</td>
<td>23.64</td>
</tr>
<tr>
<td>CD</td>
<td>8.98</td>
</tr>
</tbody>
</table>

The exact values of the effective length factors $K$ can be computed by using these multipliers and Eqs. (2) and (8). The factors used in the following comparisons are computed in this manner.

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3.2 Effective Lengths of Typical Frame Columns According to Design Codes

In this section, the “K-factor approach” used in several design codes will be applied to typical frames, the results will be compared with exact values and the errors will be determined. First, frame type AS shown in Fig. 3 will be considered. Calculating the distribution factors at both ends of the columns and using the nomograph given for non-sway frames in TS 648 reveals K factors, which are shown in column 3 of Table 2. The exact values of K factors and relative errors for this example are shown in other columns of the table.
Table 2: Effective length multipliers for frame type AS

<table>
<thead>
<tr>
<th>Storey</th>
<th>n</th>
<th>K (TS 648)</th>
<th>K (Exact)</th>
<th>Relative Error (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>1.00</td>
<td>0.813</td>
<td>0.828</td>
<td>-1.8</td>
</tr>
<tr>
<td>4</td>
<td>1.00</td>
<td>0.855</td>
<td>0.828</td>
<td>3.3</td>
</tr>
<tr>
<td>3</td>
<td>1.00</td>
<td>0.855</td>
<td>0.828</td>
<td>3.3</td>
</tr>
<tr>
<td>2</td>
<td>1.00</td>
<td>0.855</td>
<td>0.828</td>
<td>3.3</td>
</tr>
<tr>
<td>1</td>
<td>1.00</td>
<td>0.656</td>
<td>0.828</td>
<td>-20.8</td>
</tr>
</tbody>
</table>

It must be noticed that the frame under consideration has been chosen as regular as possible; hence the errors are not high from the designer’s point of view. However; it is rather interesting to encounter a quite high and unsafe error at the lowermost storey even for this example.

The K factors for all the typical frames are found in the same manner and the results are compared with the exact values. The K factors have also been found by using the formulae of TS 500 and the charts of Eurocode 3 (DIN 18800). The error ranges found for all the considered codes are shown in the Table 3.

Table 3: Error ranges of effective length factors for typical frames (%)

<table>
<thead>
<tr>
<th>Type</th>
<th>Code</th>
<th>TS 648 (AISC)</th>
<th>TS 500 (ACI)</th>
<th>Eurocode 3 (DIN 18800)</th>
</tr>
</thead>
<tbody>
<tr>
<td>AS</td>
<td></td>
<td>-20.8 ~ 3.3</td>
<td>-3.4 ~ 2.6</td>
<td>-29.6 ~ -11.0</td>
</tr>
<tr>
<td>AD</td>
<td></td>
<td>-35.0 ~ 36.7</td>
<td>-32.1 ~ 43.8</td>
<td>-51.8 ~ -3.6</td>
</tr>
<tr>
<td>BS</td>
<td></td>
<td>-43.5 ~ 9.0</td>
<td>-28.9 ~ 15.8</td>
<td>-48.6 ~ -1.2</td>
</tr>
<tr>
<td>BD</td>
<td></td>
<td>-25.7 ~ 4.9</td>
<td>-21.0 ~ 20.5</td>
<td>-32.6 ~ -2.6</td>
</tr>
<tr>
<td>CS</td>
<td></td>
<td>-40.5 ~ 14.7</td>
<td>-25.2 ~ 21.9</td>
<td>-45.9 ~ 5.6</td>
</tr>
<tr>
<td>CD</td>
<td></td>
<td>-45.1 ~ 36.7</td>
<td>-42.7 ~ 21.4</td>
<td>-51.4 ~ 0.3</td>
</tr>
<tr>
<td>Limits</td>
<td></td>
<td>-45.1 ~ 36.7</td>
<td>-42.7 ~ 43.8</td>
<td>-51.8 ~ 5.6</td>
</tr>
</tbody>
</table>

It is clearly seen that all the considered codes yield errors, which are almost of the same order reaching as high as -50%. This is due to the fact that all codes use similar formulae, which are based on the isolated subassembly approach considering only the local stiffness.
distributions. However, investigations carried on a number of numerical examples have shown that, buckling length multipliers are dependent on various factors such as
- Axial force distribution,
- Overall stiffness distribution of columns and beams,
- Location of the individual column within the frame
together with local stiffness distributions. It is concluded that, the buckling length multipliers should be determined by taking into account all these factors i.e. considering not only the local stiffness distributions, but also the overall characteristics of the structure. In the following, a simple correction method which approximately takes into account all these characteristics will be outlined and applied to numerical examples.

4. A SIMPLIFIED CORRECTION PROCEDURE
The value for the buckling load can be expressed as

\[ P_k \equiv \frac{\pi^2EI}{h_k^2} \]  

(9)

for any column by using Eq. (2). Here the indices ij are not shown for the sake of simplicity. If \( h_k \) is substituted by the expression given by Eq. (6)

\[ P_k \equiv \frac{\pi^2EI}{K^2h_k^2} \]  

(10)

is obtained for the approximate value of the buckling load. However, since the K factors taken from the codes are erroneous, the buckling loads computed by using Eq. (10) involve rather high errors as well. In addition, the values obtained by this formula are different for each column. However, investigations on several numerical examples have shown that, the average of \( P_k \) values obtained for individual columns is sufficiently close to the exact value. It is also observed that the results are further improved if a weighted average is computed by taking the parameter \( n \) for each column as weighting factor. Thus the approximate buckling load of the frame can be expressed as

\[ P_k \equiv \pi^2E \frac{\sum_{Columns} \frac{1}{K^2h_k^2}}{\sum_{Columns} n} . \]  

(11)

Once the approximate value for \( P_k \) is computed by using Eq. (11) the modified values for K factors for each individual column can readily be calculated by
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\[ K = \frac{\pi}{h_c} \sqrt{\frac{EI}{nP_k}} \]  

which is obtained from Eqs. (2) and (6). By applying this procedure it is possible to improve the K factors to a certain extent. It must be pointed out that, by applying this procedure the errors for all the frame columns acquire the same value. In the following the proposed method will be applied to typical frames and the errors will be discussed.

4.1 Improved Effective Lengths of Typical Frame Columns

In this section, the effective lengths of typical frame columns computed according to several design codes will be improved using the method given above. These improved values will then be compared with the exact values and the errors will be determined. Firstly, frame AS shown in Fig. 3 is considered and the K factors shown in column 3 of Table 2 are modified. The relevant calculations are shown in Table 4.

<table>
<thead>
<tr>
<th>Storey</th>
<th>K (TS 648)</th>
<th>( \frac{1}{K^2} )</th>
<th>K (Improved)</th>
<th>K (Exact)</th>
<th>Relative Error (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>0.813</td>
<td>1.512</td>
<td>0.794</td>
<td>0.828</td>
<td>-4.2</td>
</tr>
<tr>
<td>4</td>
<td>0.855</td>
<td>1.367</td>
<td>0.794</td>
<td>0.828</td>
<td>-4.2</td>
</tr>
<tr>
<td>3</td>
<td>0.855</td>
<td>1.367</td>
<td>0.794</td>
<td>0.828</td>
<td>-4.2</td>
</tr>
<tr>
<td>2</td>
<td>0.855</td>
<td>1.367</td>
<td>0.794</td>
<td>0.828</td>
<td>-4.2</td>
</tr>
<tr>
<td>1</td>
<td>0.656</td>
<td>2.327</td>
<td>0.794</td>
<td>0.828</td>
<td>-4.2</td>
</tr>
<tr>
<td>Total</td>
<td>7.940</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Since the values \( h_c, I \) and \( n \) are same for all the columns of this example it suffices to use the simple average of \( \frac{1}{K^2} \) in applying Eq. (11) for this example. Hence, the approximate value for the frame buckling load can be computed as

\[ P_k \equiv \pi^2 \frac{7.940 EI}{5 h^2} = 15.67 \frac{EI}{h^2} . \]
Inserting this value into equation (12) yield the improved K factors which are shown in 4th column of Table 4. The exact values of K factors and errors are also shown in consecutive columns of the table. As has been mentioned above, the errors for all the columns are found to be the same i.e. -4.2%.

All K factor values of typical frames are modified in the same manner and compared with the exact values. In addition, the K factors of typical frame columns obtained by both TS 500 formulae and Eurocode 3 (DIN 18800) charts have also been modified. Relative errors of modified K values for all the considered codes are summarized in Table 5.

<table>
<thead>
<tr>
<th>Type</th>
<th>Code</th>
<th>Type</th>
<th>Code</th>
<th>Type</th>
<th>Code</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>TS 648 (AISC)</td>
<td></td>
<td>TS 500 (ACI)</td>
<td></td>
<td>Eurocode 3 (DIN 18800)</td>
</tr>
<tr>
<td>AS</td>
<td>-4.2</td>
<td>AD</td>
<td>-10.1</td>
<td>BS</td>
<td>-19.8</td>
</tr>
<tr>
<td></td>
<td>4.7</td>
<td></td>
<td>-1.9</td>
<td></td>
<td>-9.5</td>
</tr>
<tr>
<td></td>
<td>-15.9</td>
<td></td>
<td>-21.1</td>
<td></td>
<td>-28.6</td>
</tr>
<tr>
<td>BD</td>
<td>-5.3</td>
<td>CS</td>
<td>-14.4</td>
<td>CD</td>
<td>-8.6</td>
</tr>
<tr>
<td></td>
<td>6.9</td>
<td></td>
<td>-4.9</td>
<td></td>
<td>1.5</td>
</tr>
<tr>
<td></td>
<td>-15.7</td>
<td></td>
<td>-24.6</td>
<td></td>
<td>-19.5</td>
</tr>
<tr>
<td></td>
<td>-19.8 ~ -4.2</td>
<td></td>
<td>-9.5 ~ 6.9</td>
<td></td>
<td>-28.6 ~ -15.7</td>
</tr>
</tbody>
</table>

The following results are found by comparing the errors given in Tables 5 and 3:

- The excessive errors in K factors have been reduced to a great extent by applying the proposed improvement procedure.
- Despite the application of the improvement procedure the errors for Eurocode 3 are rather high and negative (on the unsafe side).
- The best results after the application of the improvement procedure are obtained for the formulae of TS 500 (ACI). When using the proposed modification in practical applications, it is recommended that these formulae should be used initially.

4.2 Other Methods Used in Modifying the Effective Lengths

As has been mentioned above, other methods have also been developed for the improvement of effective lengths. The most practical methods among these may be regarded as those developed by Hellesland-Bjorhovde, [20] and Mahini-Seyyedian, [22]. These two methods have also been applied to the typical frames shown in Fig. 3 and the resulting errors are summarized in Table 6.
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Table 6: Relative errors for other methods (%)

<table>
<thead>
<tr>
<th>Type</th>
<th>Hellesland-Bjorhovde</th>
<th>Mahini-Seyyedian</th>
</tr>
</thead>
<tbody>
<tr>
<td>AS</td>
<td>-2.1</td>
<td>9.8</td>
</tr>
<tr>
<td>AD</td>
<td>-10.4</td>
<td>7.4</td>
</tr>
<tr>
<td>BS</td>
<td>-11.3</td>
<td>15.4</td>
</tr>
<tr>
<td>BD</td>
<td>-4.8</td>
<td>2.1</td>
</tr>
<tr>
<td>CS</td>
<td>-8.9</td>
<td>21.5</td>
</tr>
<tr>
<td>CD</td>
<td>-8.4</td>
<td>10.8</td>
</tr>
<tr>
<td>Limits</td>
<td>-11.3 ~ -2.1</td>
<td>2.1 ~ 21.5</td>
</tr>
</tbody>
</table>

It is seen that all the error ratios for the method of Hellesland-Bjorhovde are negative (on the unsafe side). On the other hand, the error ratios for the method of Mahini-Seyyedian are all positive (on the safe side). When these error ratios are compared with the ones in Table 5, it is observed that the errors for TS 500 K factors are generally smaller. Moreover, the total computational amount is lower for the proposed improvement method.

5. CONCLUSIONS

In this paper, a simple improvement method is developed for determining the effective lengths of columns of braced frames and the application of the method on numerical examples is explained. The main conclusions obtained in this study may be stated as follows:

1. It is shown that simplified formulae, nomographs or charts given in several design codes for determining the effective lengths of frame columns may yield results with excessive errors even for some regular structures. This is mainly due to the fact that, the codes adopt the isolated subassembly approach; hence ignore the general behaviour of the system.

2. A significant portion of the effective lengths found by using the design codes may have negative errors up to -50% i.e. the effective lengths are smaller than the exact values, hence are on the unsafe side. It is clear that this is a rather undesirable situation in practical applications.

3. In this paper, a simple method has been presented for improving the effective lengths of braced frame columns. The effective lengths of columns are determined after computing the approximate frame buckling load.

4. The approximate value of the frame buckling load is determined by taking a weighted average of the effective lengths of columns found by using the design codes. It is found that, the error ratios for the proposed improvement method are lower than 10%
especially when the K factor coefficients of TS 500 (ACI) are used initially. It may be stated that the error ratios of this order are quite acceptable for practical applications.

References

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[29] SAP2000-V8, Structural Analysis Program, CSI, Berkeley, USA.